ELECTRIC DISCHARGE IN A JET OF CONDUCTING VISCOUS FLUID

L. A. Vulis and K. E. Dzhaugashtin

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The effect of a jet flow on an electric discharge is demonstrated within the framework of the theory of the magnetic boundary layer of the second kind at Reynolds numbers $R_m \gg 1$.

1. In [1] V. N. Zhigulev showed that the interaction of a magnetic field induced by a discharge current with a moving conducting medium is localized at magnetic Reynolds numbers $R_m \gg 1$ in a narrow region of a unique magnetic boundary layer. If the vector **H** is orthogonal to the plane of the velocity vector **V** (magnetic boundary layer of the second kind [1]), then, as has been shown in another paper by the same author [2], an electric discharge in the homogeneous flow of an ideal fluid converges toward the axis, assuming the form of an "electric jet." In the same paper it was shown that this effect may be used to obtain high temperatures, and a solution of a corresponding thermal problem was included.

In [3, 4] there were examined several special problems concerning the motion of a viscous conducting fluid (a jet flow and a homogeneous flow) along a body at the surface of which an electric discharge occurs.

Beside the aspect (of obtaining high temperatures), examined in [2], of the problem of a discharge in the flow of a conducting medium, the effects of the hydrodynamics of the flow on the form and nature of the electric discharge are also of interest.

In this connection we shall examine some of the simplest problems concerning the form of a discharge in free laminar jet flows, solutions of which may be obtained within the framework of the theory of the magnetic boundary layer of the second kind.

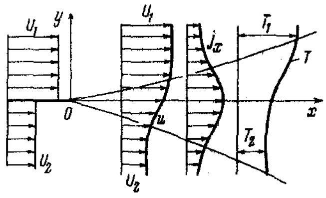

Fig. 1

Assuming, for simplicity, that $\rho = \text{const}$, and neglecting the temperature dependence of conductivity, it is possible to use for the velocity profile the solution of the corresponding self-similar problem.

In this case, the effect of the magnetic field on the motion is limited to the creation of a static-pressure gradient, the total pressure (i.e., the sum $P^+ = p + p_m = p + 1/2\mu H^2$) remaining constant.

Having introduced these limitations, we may speak of two effects: a rearrangement of the electric lines of force under the effect of the motion of the medium, and a rearrangement of the temperature field under the effect of Joule dissipation.

The initial equations for the plane steady flow of a viscous incompressible fluid we write in the form

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}, \quad \frac{\partial P^+}{\partial y} = 0, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

$$u\frac{\partial H}{\partial x} + v\frac{\partial H}{\partial y} = \nu_m\frac{\partial^2 H}{\partial y^2}, \quad j_x = \frac{\partial H}{\partial y}, \quad j_y = \frac{\partial H}{\partial x},$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = a\frac{\partial^2 T}{\partial y^2} + \frac{\nu}{C_p}\left(\frac{\partial u}{\partial y}\right)^2 + \frac{\nu_m}{\rho C_p}\left(\frac{\partial H}{\partial y}\right)^2. \quad (1.1)$$

2. As a first example, we shall examine the problem of a discharge that takes place in the boundary layer at the edge of a plane jet. Let (see Fig. 1) a uniform flow of a viscous incompressible fluid run off a flat plate into a medium at rest or one that moves in the manner of a wake flow. Starting from the edge of the plate, a boundary layer forms in the flow, which possesses a characteristic velocity profile u(y), as shown in Fig. 1. Let us assume that between the edge of the plate and a region very distant from it there is a constant potential difference. In the absence of motion, the electric charge would closely resemble a cylindrical one (the edge of the plate that forms the O_z axis is the symmetry axis).

In the presence of jet-flow motion, the velocity profile is described by the known formula [5]

$$\frac{u}{u_1} = F'(\varphi) = 1 + \frac{1}{2}\left(\frac{u_2}{u_1} - 1\right)(1 - \operatorname{erf}\varphi),$$

$$\varphi = \frac{1}{2}\left(\frac{u_1}{\nu x}\right)^{1/2}\frac{y}{x}. \quad (2.1)$$

When

$$H = H_\infty,\ T = T_1 \quad \text{at} \quad y = +\infty,$$
$$H = -H_\infty,\ T = T_2 \quad \text{at} \quad y = -\infty \quad (2.2)$$

are taken as the boundary conditions for the magnetic field and temperature, the similar solutions

$$\frac{H + H_\infty}{2H_\infty} = h(\varphi), \qquad \frac{T - T_2}{T_1 - T_2} = \Theta(\varphi) \quad (2.3)$$

of the ordinary differential equations

$$h'' + 2P_m F h' = 0,$$

$$\Theta'' + 2PF\Theta' = S\frac{P}{P_m}(h')^2 \qquad \left(S = \frac{4H_\infty^2}{\rho C_p (T_1 - T_2)}\right) \quad (2.4)$$

with the boundary conditions

$$h = \Theta = 1 \quad \text{at} \quad \varphi = +\infty,$$
$$h = \Theta = 0 \quad \text{at} \quad \varphi = -\infty, \quad (2.5)$$

may be written in the form

$$h(\varphi) = \Big[\int_{-\infty}^{\infty} (F''_{(\varphi)})^{P_m} d\varphi\Big]^{-1} \Big[\int_{-\infty}^{\varphi} (F''(\varphi))^{P_m} d\varphi\Big],$$

$$\Theta(\varphi) = \Theta_0 + S(1-\Theta_0)\Theta_1, \tag{2.6}$$

$$\Theta_0(\varphi) = \Big[\int_{-\infty}^{\infty} (F''(\varphi))^{P} d\varphi\Big]^{-1} \Big[\int_{-\infty}^{\varphi} (F''(\varphi))^{P} d\varphi\Big],$$

$$\Theta_1(\varphi) = \frac{P}{P_m}\int_{-\infty}^{\varphi} (F''(\varphi))^{P} \Big[\int_0^{\varphi} (h'(\varphi))^2 (F''(\varphi))^{-P} d\varphi\Big] d\varphi,$$

where P and P_m are the hydrodynamic and magnetohydrodynamic Prandtl numbers, respectively.

As an illustration, Fig. 2 shows the computation results for a simple example, where the curve 1 yields F'/P, the curves 2, 3, and 4 correspond to the values of $P_m = 1$, 0.5, and 0.25, and the curve 5 corresponds to P = 1 for $\sigma \sim T$. It may be seen that, just as in [2], the current profile has a symmetric streamline shape, while the temperature profile is asymmetric.

If the temperature dependence of the conductivity is taken into account, then, for an incompressible fluid, the problem reduces to the integral equation

$$\Theta(\varphi) = \int_{-\infty}^{\varphi} \frac{1}{\Phi(\varphi)}\Big[SP\int_{-\infty}^{\varphi} \sigma(T)(h'(\varphi))^2\Phi(\varphi) + C\Big] d\varphi,$$

$$\Phi(\varphi) = \exp 2P\varphi \int_{-\infty}^{\varphi} F\Theta' d\varphi. \tag{2.7}$$

Here, the constant C is defined by the expression

$$C = 1 - \Big[SP\int_{-\infty}^{\infty}\frac{1}{\Phi(\varphi)}\Big(\int_{-\infty}^{\varphi}\sigma(T)(h'(\varphi))^2\Phi(\varphi)d\varphi\Big)d\varphi\Big]\times$$

$$\times\Big[\int_{-\infty}^{+\infty}\frac{d\varphi}{\Phi(\varphi)}\Big]^{-1}, \quad h(\varphi) = \Big[\int_{-\infty}^{\infty}\sigma(T)\exp\Big(-2P\int_0^{\varphi}F\sigma d\varphi\Big)d\varphi\Big]^{-1}\times$$

$$\times\Big[\int_{-\infty}^{\varphi}\sigma(T)\exp\Big(-2P\int_0^{\varphi}F\sigma\, d\varphi\Big)d\varphi\Big].$$

A (numerical) solution for a specific dependence of $\sigma(T)$ leads to a still greater narrowing of such a region. Here a peculiar avalanche-type mechanism takes effect—in the region where the current passes, the temperature, and with it the conductivity, increases, which in turn leads to an increase in current, and so forth. An idea of this is given by the dashed curve in Fig. 2 [in the case of a linear dependence of $\sigma(T)$].

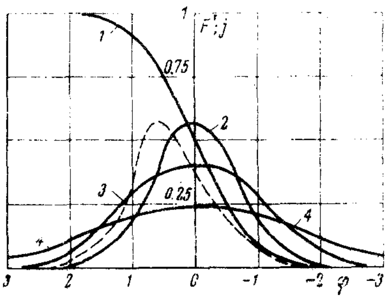

Fig. 2

3. The problem of a plane jet source is solved in the same fashion. The solutions of the equations of this problem must satisfy the boundary conditions

$$\begin{gathered}\text{at } y = 0,\\ du/dy = 0, \quad v = 0, \quad H = 0;\\ \text{at } y = \pm\infty,\\ u = 0, \quad H = \pm H_\infty.\end{gathered} \tag{3.1}$$

The velocity distribution in the jet in this case has the form

$$\frac{u}{u_m} = F'(\varphi) = \frac{1}{\mathrm{ch}^2\varphi},$$

$$u_m = \frac{1}{2}\Big(\frac{3I_x^2}{4\rho^2\nu_x}\Big)^{1/3}, \quad \varphi = \frac{1}{2}\Big(\frac{I_x}{6\rho\nu^2x^2}\Big)^{1/3} y,$$

$$I_x = \int_{-\infty}^{\infty}\rho u^2 = \text{const}. \tag{3.2}$$

Introducing the condition that the solution is nontrivial,

$$\int_0^{\infty} uH dy = K \neq 0, \tag{3.3}$$

we obtain for the magnetic field a differential equation

$$h'' + 2P_m F h' = 0 \qquad (h(+\infty) = 1,\ h(-\infty) = -1). \tag{3.4}$$

The solution has the form

$$h(\varphi) = 2\Big[\int_{-\infty}^{\infty}(F'(\varphi))^{P_m}d\varphi\Big]^{-1}\Big[\int_{-\infty}^{\varphi}(F'(\varphi))^{P_m}d\varphi\Big] - 1. \tag{3.5}$$

As for the energy equation, we shall solve it for two variants of the boundary conditions for the temperature: symmetric (A) and nonsymmetric (B) with respect to velocity.

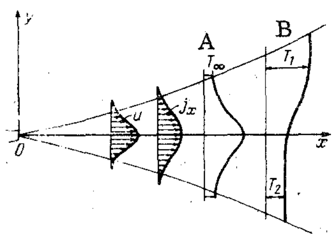

Fig. 3

Case A

$$\begin{gathered}\frac{\partial T}{\partial y} = 0 \quad \Big(\frac{\partial u}{\partial y} = 0\Big) \quad \text{at } y = 0,\\ T = T_\infty \quad (u = 0) \quad \text{at } y = \pm\infty.\end{gathered} \tag{3.6}$$

We shall write the energy equation in the form

$$u\frac{\partial\Pi}{\partial x} + v\frac{\partial\Pi}{\partial y} = \nu\frac{\partial^2\Pi}{\partial y^2} + \nu\Big(\frac{1}{P} - 1\Big)\frac{\partial^2 u}{\partial y^2} + \nu_m\Big(\frac{1}{P_m} - 1\Big)\frac{\partial^2 H}{\partial y^2},$$

$$\Big(\Pi = i - i_\infty + \frac{u^2}{2} + \frac{\mu}{\rho}(H^2 - H_\infty^2)\Big) \tag{3.7}$$

For Prandtl numbers $P = P_m = 1$, the equation (3.7) admits a simple integral, analogous to the Crocco integral in conventional gasdynamics,

$$\Pi = C_1 u + C_2 . \qquad (3.8)$$

Taking the boundary conditions into account, we have

$$i = i_\infty + \frac{A}{I_0} 0.454 \left(\frac{I_0^2}{\rho \nu x}\right)^{1/3} \mathrm{ch}^{-2}\varphi - 0.103 \left(\frac{I_0^2}{\rho^2 \nu x}\right)^{2/3} \times$$
$$\times \mathrm{ch}^{-4}(\varphi) - \frac{\mu H_\infty^2}{2\rho}[(h(\varphi))^{-2} - 1] \qquad (3.9)$$
$$A = \int_{-\infty}^{\infty} u \Pi dy.$$

The velocity, current, and temperature profiles will be symmetrical, with maxima on the jet axis.

Case B

$$T = T_1 \quad \text{at} \quad y = +\infty, \qquad T = T_2 \quad \text{at} \quad y = -\infty. \qquad (3.10)$$

Postulating

$$\frac{T - T_2}{T_1 - T_2} = \Theta(\varphi), \qquad (3.11)$$

we get the equation

$$\Theta'' + 2PF\Theta' + \frac{P}{P_m} S (h')^2 = 0,$$
$$(\Theta(+\infty) = 1, \ \Theta(-\infty) = -1). \qquad (3.12)$$

The solution has the form

$$\Theta(\varphi) = \Theta_0(\varphi) + S(\Theta_0\Theta_1(\infty) - \Theta_1(\varphi)),$$
$$\Theta_0(\varphi) = \left[\int_{-\infty}^{\infty} (F'(\varphi))^P dy\right]^{-1} \left[\int_{-\infty}^{\varphi} (F'(\varphi))^P d\varphi\right], \qquad (3.13)$$
$$\Theta_1(\varphi) = \frac{P}{P_m} \int_{-\infty}^{\infty} (F'(\varphi))^P \left\{\int_{-\infty}^{\infty} (F'(\varphi))^{-P} (h'(\varphi))^2 d\varphi\right\} d\varphi.$$

In this case both current and velocity profiles will be symmetrical, while the temperature profile will resemble the corresponding profile for the edge of the jet. Figure 4 gives the results of computations for $P = P_m = 1$, where the curves 1, 2, 3, 4, 5 correspond to the values $S = 0, 1, 2, 3, 4$.

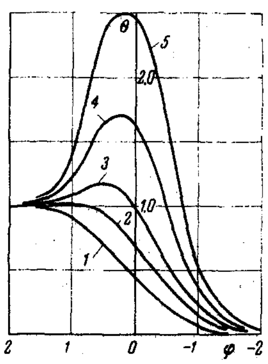

Fig. 4

4. Let us also determine the magnetic field and current distribution for an axisymmetric jet source having a velocity profile that corresponds to the expression

$$\frac{u}{u_m} = \frac{F'(\varphi)}{\varphi} = \frac{1}{(1 + {}^1/_8\varphi^2)^2},$$
$$u_m = \frac{3I_x}{8\pi\rho\nu x}, \quad \varphi = \left(\frac{3I_x}{8\pi\rho\nu^2 x^2}\right)^{1/2} y, \qquad (4.1)$$
$$I_x = 2\pi \int_0^\infty y\rho u^2 dy = \text{const}.$$

The initial equation

$$u\frac{\partial H}{\partial x} + v\frac{\partial H}{\partial y} = \nu_m \frac{1}{y}\frac{\partial}{\partial y}\left(y\frac{\partial H}{\partial y}\right), \qquad (4.2)$$

with the boundary conditions

$$H = 0 \quad \text{at} \quad y = 0, \quad H = H_\infty \quad \text{at} \quad y = \infty \qquad (4.3)$$

admits a similar solution of the equation

$$(\varphi h')' + P_m (Fh)' = 0 \qquad (4.4)$$

in the form

$$h(\varphi) = \left(\frac{F'}{\varphi}\right)^{P_m} = \frac{1}{(1 + {}^1/_8\varphi^2)^{2P_m}}. \qquad (4.5)$$

The energy equation of this problem is not integrable in quadratures, even for $P = 1$.

It should be noted that analogous solutions might be obtained for turbulent jet flows if one proceeds from conventional semi-empirical schemes. Qualitatively, the results would be the same; quantitatively, a wider turbulent jet would lead to stronger compression of the current sheet. However, the validity of an extension of conventional computation methods (e.g., the displacement path theory, and other methods) to include the motion of a conducting medium in the range of large magnetic Reynolds numbers R_m requires special verification.

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